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An inequality relating total mass and the area of a trapped surface in general relativity

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Abstract. Let M be a space-time which is asymptotically flat at past null infinity $\mathscr{I}^-$, satisfies the dominant energy condition and contains a trapped surface $\mathscr{T}$. We show that if $\mathscr{T}$ can be connected to $\mathscr{I}^-$ by means of a non-singular null-hypersurface $\mathcal{N}$, then $m^2 \geqslant A/16\pi$ where m is the Bondi mass with respect to $\mathcal{N}$ and A in the area of $\mathscr{T}$.

A well known consequence of some suitable assumption of cosmic censorship such as asymptotic predictability (Hawking and Ellis 1973) together with the so-called 'no-hair' conjecture/theorem of Israel-Carter-Hawking-Robinson (Carter 1979), which asserts that a black hole will settle down into a Kerr-Newman space-time, is that the total mass of an isolated gravitating system satisfies the inequality

$$16\pi m^2 \geqslant A \tag{1}$$

where A is the area of some trapped surface $\mathscr{T}$. (For an outline of the argument see e.g. Penrose (1982).) Since the most doubtful physical assumption used in the proof of (1) is cosmic censorship, a direct proof of (1) may be regarded as giving some sort of test for the censorship hypothesis (Gibbons 1972).

In this paper we shall present a simple and direct proof of this inequality which does not use cosmic censorship. We use a Witten type argument similar to that used in proving the positivity of the Bondi mass (Ludvigsen and Vickers 1982, Horowitz and Perry 1982) except that our spinor propagation law is based upon a null hypersurface. This greatly simplifies the analysis as spinor methods are much more natural on such surfaces. The use of null hypersurfaces in the present context does, however, have the drawback that it leads to inequality (1) where m is the Bondi mass with respect to past null infinity $\mathscr{I}^-$ and, in order for this mass to be well defined, we must assume a certain degree of asymptotic flatness at $\mathscr{I}^-$. Furthermore, if we wish (1) to hold for the ADM mass at space-like infinity, i^0, we must assume a certain degree of regularity in the region of i^0, in which case the Bondi mass-gain formula on $\mathscr{I}^-$ gives (Ashtekar and Magnon-Ashtekar 1979)

$$m_{\mathrm{ADM}} \geqslant m_{\mathrm{B}} \tag{2}$$

and hence

$$16\pi m^2_{\mathrm{AMD}} \geqslant A. \tag{3}$$

Our main result is the following:

Theorem. Let M be a space–time which is asymptotically flat at $\mathscr{I}^-$, which satisfies the dominant energy condition and contains a 'convex' trapped surface $\mathscr{T}$ (homeomorphic to S^2) which may be connected to $\mathscr{I}^-$ by means of a non-singular null hypersurface $\mathscr{N}$. Then $16\pi m \geqslant A$, where m is the Bondi mass at the advanced time defined by $\mathscr{N}$ and A is the area of $\mathscr{T}$. (See note below for definition of 'convex'.)

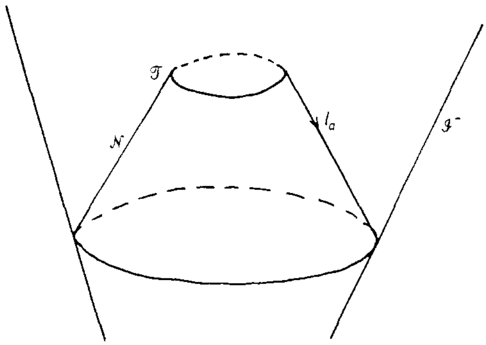

Figure 1.

Since the proof of the theorem in the vacuum case is very similar to that in the non-vacuum case we shall, for the sake of simplicity and brevity, deal with the vacuum case. Details of our proof in the non-vacuum case will be given elsewhere.

We start by introducing a suitable coordinate system (v, r, θ, ϕ), in a neighbourhood of $\mathscr{N}$ together with an associated Geroch–Held–Penrose (1973) (GHP) spinor dyad. Let v be an advanced time parameter which is chosen so that $\mathscr{N}$ is given by $v = 0$, let r be an affine parameter along the null geodesic generators of the hypersurfaces $v = \text{constant}$ and finally let θ and ϕ be constant along such generators. Now let o_A be chosen so that (using the Penrose (1968) abstract index notation) $o_A \iota_{A'} = l_a = \nabla_a v$, $\text{þ} o_A = 0$ and ι_A be chosen so that $o_A \iota^A = 1$ and $n_a = \iota_A \iota_{A'}$ is orthogonal to the two-surface $S_r := \{x \in \mathscr{N} : r = \text{constant}\}$. For such a spinor dyad we have (Hawking 1968)

$$\kappa = \varepsilon = 0, \qquad \rho = \bar{\rho}, \qquad \rho' = \bar{\rho}', \qquad \tau = -\bar{\tau}' = \beta - \bar{\beta}', \tag{4}$$

and for a quantity η of weight (p, q) with $p - q = -2$ we have

$$\oint_{S_r} \eth \eta \, \mathrm{d}S_r = -\tfrac{1}{2}(p+q) \oint_{S_r} \tau \eta \, \mathrm{d}S_r \tag{5}$$

which reduces to the usual result for quantities intrinsic to S_r which have zero boost weight (i.e. $p + q = 0$). We now specialise to an affine parameter which is chosen so that:

(i) $r = r_0 = \text{constant}$ on $\mathscr{T}$;

(ii) $\rho = 1/r + \mathrm{O}(r^{-3})$ (N.B. no r^{-2} term);

(iii) $\lim_{r\to\infty} r^{-2} \oint_{S_r} \mathrm{d}S_r = 4\pi$.

Such an r always exsists, but is not in general a Bondi radial coordinate since S_r need not tend to a metric sphere†. Let $\mathrm{d}S_r = \mathrm{d}S_0 / V^2$ where $V > 0$ and $\mathrm{d}S_0$ is the surface

† *Note*: We will however assume that $\mathscr{T}$ is 'convex' in the sense that, asymptotically, S_r has positive Gaussian curvature.

element of a unit sphere, and let $V_0 := \lim_{r\to\infty} rV$. Then

$$4\pi = \lim_{r\to\infty} r^{-2} \oint_{S_r} \mathrm{d}S_r = \lim_{r\to\infty} \oint_{S_r} \frac{\mathrm{d}S_r}{r^2 V^2} = \oint \frac{\mathrm{d}S_0}{V_0^2}. \tag{6}$$

Since $\mathscr{T}$ is a trapped surface we have (Penrose 1968)

$$\rho \leqslant 0 \qquad \text{and} \qquad \rho' \leqslant 0 \qquad \text{on } \mathscr{T}. \tag{7}$$

Furthermore by equation (2.22) of GHP we have

$$Þ\rho = \rho^2 + \sigma\bar{\sigma} \geqslant 0 \tag{8}$$

so that by (8) and (ii) above we must have $\rho < 0$ over the whole of $\mathscr{N}$.

We now introduce two spinor fields λ_A and μ_A on $\mathscr{N}$ which are chosen to satisfy the propagation equations

$$Þ\lambda_0 = 0, \tag{9}$$

$$ð'\lambda_0 + \rho\lambda_1 = 0, \tag{10}$$

$$Þ\mu_0 = 0, \tag{11}$$

$$ð'\mu_0 + \rho\mu_1 = 0, \tag{12}$$

together with the asymptotic conditions

$$\lim_{r\to\infty} r\, ð\lambda_0 = 0, \qquad \lim_{r\to\infty} r\, ð\mu_o = 0, \tag{13}$$

$$\lambda_0\lambda_{0'} + \mu_0\mu_{0'} = 1/V_0. \tag{14}$$

It is easy to see that such a pair of spinor fields exists and is well defined on $\mathscr{N}$.

We now define

$$I(r) := -(4\pi)^{-1} \oint_{S_r} [\rho(\lambda_1\lambda_{1'} + \mu_1\mu_{1'}) + \rho'(\lambda_0\lambda_{0'} + \mu_0\mu_{0'})]\, \mathrm{d}S_r. \tag{15}$$

By using the asymptotic expansions of ρ and ρ' appropriate to our affine parameter and spinor dyad (obtained from those in Exton *et al* (1969) by making the null rotation $\iota_A \to \iota_A - \bar{\omega} o_A$ and by rescaling r) it may be shown that the Bondi mass is given by

$$m = \lim_{r\to\infty} I(r). \tag{16}$$

After a fairly lengthy but straightforward spin-coefficient calculation one can show

$$\frac{\mathrm{d}I}{\mathrm{d}r} = \frac{1}{4\pi} \oint_{S_r} (X\bar{X} + Y\bar{Y})\, \mathrm{d}S_r \qquad \text{where } X = ð\lambda_0 + \sigma\lambda_1, \qquad Y = ð\mu_0 + \sigma\mu_1, \tag{17}$$

and thus

$$4\pi m = \int_{r=r_0}^{\infty} \oint_{S_r} (X\bar{X} + Y\bar{Y})\, \mathrm{d}S_r\, \mathrm{d}r$$

$$- \oint_{S_{r_0}} [\rho(\lambda_1\lambda_{1'} + \mu_1\mu_{1'}) + \rho'(\lambda_0\lambda_0 + \mu_0\mu_{0'})]\, \mathrm{d}S_{r_0} \tag{18}$$

$$\geqslant \int_{r=r_0}^{\infty} \oint_{S_r} (X\bar{X} + Y\bar{Y})\, \mathrm{d}S_r\, \mathrm{d}r - \oint_{S_{r_0}} \rho(\lambda_1\lambda_{1'} + \mu_1\mu_{1'})\, \mathrm{d}S_{r_0} =: \tilde{I}(r_0) \tag{19}$$

since $\rho' \leqslant 0$ on the trapped surface $\mathscr{T} = S_{r_0}$.

Now a fairly easy calculation gives

$$\mathrm{d}\tilde{I}/\mathrm{d}r = \oint_{S_r} (\bar{\eth}\lambda_0 \eth\lambda_{0'} - \eth\lambda_0 \bar{\eth}\lambda_{0'} - \tau'\lambda_0 \eth\lambda_{0'} - \bar{\tau}'\lambda_{0'} \bar{\eth}\lambda_0$$

$$+ \bar{\eth}\mu_0 \eth\mu_{0'} - \eth\mu_0 \bar{\eth}\mu_{0'} - \tau'\mu_0 \eth\mu_{0'} - \tau'\mu_{0'} \bar{\eth}\mu_0)\, \mathrm{d}S_r \tag{20}$$

On integrating by parts, using equations (5), (14) and equation (3.12) of GHP

$$(\bar{\eth}\eth - \eth\bar{\eth})\eta = -(pK - q\bar{K})\eta \tag{21}$$

where

$$K = \sigma\sigma' + \rho\rho' - \psi_2 \tag{22}$$

together with the fact that

$$\eth(\lambda_o\lambda_{o'} + \mu_0\mu_{o'}) = \eth(1/V_0) = -\tau/V_0. \tag{23}$$

We obtain

$$\frac{\mathrm{d}\tilde{I}}{\mathrm{d}r} = \oint_{S_r} \frac{1}{2}\frac{(K+\bar{K})}{V_0}\,\mathrm{d}S_r - \oint_{S_r} \frac{\tau\bar{\tau}}{V_0}\,\mathrm{d}S_r \tag{24}$$

$$\geqslant \frac{1}{2}\oint_{S_r} \frac{(K+\bar{K})}{V_0}\,\mathrm{d}S_r =: J(r). \tag{25}$$

A straightforward spin-coefficient calculation gives

$$\mathrm{d}J/\mathrm{d}r = 0 \tag{26}$$

so that

$$\frac{\mathrm{d}\tilde{I}}{\mathrm{d}r} \geqslant \lim_{r\to\infty} \oint_{S_r} \frac{1}{2}\frac{(K+\bar{K})}{V_0} =: B. \tag{27}$$

On the other hand the asymptotic flatness conditions at $\mathscr{I}^-$ give $\tilde{I}(r) = Br + \mathrm{O}(r^{-1})$ (i.e. no constant term) so we must have

$$\tilde{I}(r)/r \geqslant B \tag{28}$$

and thus

$$(4\pi m)^2/r_0^2 \geqslant B^2. \tag{29}$$

Now $R = K + \bar{K}$ is the Gaussian curvature of S_r so that

$$B = \frac{1}{2}\oint \frac{\mathring{R}}{V_0^3}\,\mathrm{d}S_0 \qquad \text{where } \mathring{R} = \lim_{r\to\infty} r^2 R. \tag{30}$$

Using equation (A7) of Newman and Tod (1977) it may be shown that

$$\mathring{R} = V_0^2(1 - \nabla^2 \ln(1/V_0)) \tag{31}$$

where ∇^2 is the Laplacian on the metric sphere so that

$$B = \tfrac{1}{2}\int V_0^{-1}[1 - \nabla^2 \ln(1/V_0)]\,\mathrm{d}S_0 \tag{32}$$

and therefore

$$16\pi^2 m^2/r_0^2 \geqslant \left(\tfrac{1}{2}\oint V_0^{-1}[1-\nabla^2 \ln(1/V_0)]\, dS_0\right)^2 \tag{33}$$

$$\geqslant \pi \oint dS_0/V_0^2 = 4\pi^2 \tag{34}$$

by the inequality (23) of Penrose (1982) (the condition that $1-\nabla^2 \log(1/V_0)>0$ being guaranteed by our equation (31) and our convexity condition).

We have thus shown that

$$4m^2 \geqslant r_0^2. \tag{35}$$

We now proceed to show that $4\pi r_0^2 \geqslant A$, and complete the proof. By (7) we have $r+\rho^{-1}=O(r^{-1})$ and by (8), $Þ(r+\rho^{-1})=-\sigma\bar{\sigma}/\rho^2 \leqslant 0$, and thus $r+\rho^{-1}\geqslant 0$. Hence

$$r\rho + 1 \leqslant 0 \qquad (\text{since } \rho < 0 \text{ on } \mathcal{N}). \tag{36}$$

Now

$$Þ V = \rho V \qquad \text{and} \quad rV = V_0 + O(r^{-1}) \tag{37}$$

so that $Þ(rV) = V(1+\rho r) \leqslant 0$ by the above, and hence, by (37), $rV \geqslant V_0$ and thus

$$r^2 \geqslant V_0^2/V^2. \tag{38}$$

Finally we have

$$4\pi r_0^2 = \oint r_0^2\, dS_0/V_0^2 \qquad \text{by (6)}$$

$$\geqslant \oint_{S_{r_0}} dS_0/V^2 = \oint dS_{r_0} = A$$

so that

$$4\pi r_0^2 \geqslant A \tag{39}$$

and thus by (35)

$$16\pi m^2 \geqslant r_0^2 \geqslant A \tag{40}$$

which proves the theorem.

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